Math 53: Worksheet 6 Solutions

October 12

1. Compute:

   (a) ∇f for $f(x, y, z) = x^2 \sin(yz)$.

   We have
   $$\boxed{\nabla f(x, y, z) = \langle 2x \sin(yz), x^2 z \cos(yz), x^2 y \cos(yz)\rangle}.$$

   (b) Directional derivative of $g(x, y, z) = y^2 e^x - /cos(xz)$ at $(0, -1, \pi/2)$ in the direction of $\mathbf{u} = (1, 5, -4)$.

   Note first that $\nabla g(x, y, z) = \langle y^2 e^x + z \sin(xz), 2ye^x, x \sin(xz)\rangle$ so that $\nabla g(0, -1, \pi/2) = \langle 1, -2, 0\rangle$. Next observe that the unit vector in the direction of $\mathbf{u}$ is $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{42}}\langle 1, 5, -4\rangle$. Hence,
   $$D_{\hat{\mathbf{u}}} g(0, -1, \pi/2) = \nabla g(0, -1, \pi/2) \cdot \hat{\mathbf{u}} = \boxed{-\frac{9}{\sqrt{42}}}.$$

   (c) Equation of tangent plane to the surface $xy + yz + zx + 3 = e^{xyz}$ at $(-1, 2, 0)$.

   Let $F(x, y, z) = xy + yz + zx - e^{xyz}$. The given surface is therefore a level surface of F so a vector normal to it at $P(-1, 2, 0)$ is
   $$\mathbf{n} = \nabla F|_P = \langle y + z - yze^{xyz}, x + z - xze^{xyz}, x + y - xye^{xyz}\rangle|_P = \langle 2, -1, 3\rangle.$$

   The tangent plane to the surface at P therefore is
   $$2x - y + 3z = \langle -1, 2, 0\rangle \cdot \langle 2, -1, 3\rangle \Rightarrow \boxed{2x - y + 3z = -4}.$$

2. Let $\mathbf{u} = (a, b)$ be a unit vector and let $f(x, y)$ have continuous second-order partial derivatives. Find an expression for $D_{\mathbf{u}}(D_{\mathbf{u}} f)(x, y)$.

   We have $D_{\mathbf{u}}(D_{\mathbf{u}} f)(x, y) = \nabla(D_{\mathbf{u}} f)(x, y) \cdot \mathbf{u}$. Note first that
   $$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = af_x(x, y) + bf_y(x, y).$$

   so that
   $$\nabla(D_{\mathbf{u}} f)(x, y) = \langle af_{xx}(x, y) + bf_{yx}(x, y), af_{xy}(x, y) + bf_{yy}(x, y)\rangle.$$

We conclude that

$$\begin{aligned} D_{\mathbf{u}}(D_{\mathbf{u}}f)(x,y) &= a(af_{xx}(x,y)+bf_{yx}(x,y))+b(af_{xy}(x,y)+bf_{yy}(x,y)) \\ &= a^2f_{xx}(x,y)+2abf_{xy}(x,y)+b^2f_{yy}(x,y) \end{aligned}$$

where we used Clairaut's theorem.

3. The temperature T in an infinite ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(2,1,2)$ is 60° F.

   (a) Find the rate of change of T at $(-2,1,0)$ in the direction toward the point $(4,1,-1)$.

   The temperature T (in °F) at (x,y,z) obeys $T(x,y,z) \propto \frac{1}{\sqrt{x^2+y^2+z^2}}$ so that $T(x,y,z) = \frac{k}{\sqrt{x^2+y^2+z^2}}$. Using the fact that $T(2,1,2)=60$, we get

   $$60 = \frac{k}{\sqrt{4+1+4}} \Rightarrow k = 180.$$

   Thus, $T(x,y,z) = \frac{180}{\sqrt{x^2+y^2+z^2}}$.
   Let P and Q be the points $(-2,1,0)$ and $(4,1,-1)$ respectively. We then have $\vec{PQ} = \langle 4,1,-1\rangle - \langle -2,1,0\rangle = \langle 6,0,-1\rangle$ so that $\hat{PQ} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{1}{\sqrt{37}}\langle 6,0,-1\rangle$.
   Note next that $\nabla T(x,y,z) = \left\langle \frac{-180x}{(x^2+y^2+z^2)^{3/2}}, \frac{-180y}{(x^2+y^2+z^2)^{3/2}}, \frac{-180z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$ so that $\nabla T(-2,1,0) = \left\langle \frac{360}{(5)^{3/2}}, \frac{-180}{(5)^{3/2}}, 0 \right\rangle$.
   Finally, we get

   $$D_{\hat{PQ}}T(-2,1,0) = \nabla T(-2,1,0)\cdot \hat{PQ} = \frac{2160}{(37)^{1/2}(5)^{3/2}} = \boxed{\frac{432}{\sqrt{185}}}.$$

   (b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points towards the origin.

   From (a), we have $\nabla T(x,y,z) = \left\langle \frac{-180x}{(x^2+y^2+z^2)^{3/2}}, \frac{-180y}{(x^2+y^2+z^2)^{3/2}}, \frac{-180z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$ which points in the direction opposite to $\mathbf{r} = \langle x,y,z\rangle$. The latter goes from the origin to a point (x,y,z) so $\nabla T(x,y,z)$, the direction of greatest increase in temperature, points towards the origin.

4. Consider a swimming pool with the temperature of the water at (x,y,z) given by $T(x,y,z)$. A fish swims through the water with position at time t given by $\mathbf{p}(t)$.

   (a) What is the directional derivative of T in the direction that the fish is traveling in at time 0?

At $t = 0$, the fish is traveling in the direction $\mathbf{p}'(0)$ so the required directional derivative is $\boxed{\nabla T(\mathbf{p}(0)) \cdot \dfrac{\mathbf{p}'(0)}{\|\mathbf{p}'(0)\|}}$.

(b) The fish feels a temperature at any point in time. How fast does the temperature that the fish feels change, at time 0?

The temperature felt by the fish at time t is $f(t) = T(\mathbf{p}(t))$ so that

$$f'(0) = \boxed{\nabla T(\mathbf{p}(0)) \cdot \mathbf{p}'(0)}.$$

(c) At $t = 1$, the fish decides it is happy with its current temperature. Describe/specify a set of directions (vectors) in which the fish should swim.

A direction $\mathbf{u}$ is favorable if it leads to no change in the temperature, that is, $D_{\mathbf{u}}T(\mathbf{p}(1)) = 0$. We therefore require that

$$\boxed{\nabla T(\mathbf{p}(1)) \cdot \mathbf{u} = 0}.$$

In other words, the fish could swim in any direction perpendicular to the direction of greatest temperature increase.

(d) The fish changes its mind instantaneously at time $t = 1$. It goes in the direction such that the water gets colder, fastest. Give a vector pointing in this direction.

The temperature decreases fastest in the direction $\boxed{-\nabla T(\mathbf{p}(1))}$.

5. Let $f(x, y, z, t)$ be a smooth function and let $\nabla f = \langle f_x, f_y, f_z \rangle$ be the gradient in the *space* variables only. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a smooth curve and $\mathbf{v}(t) = \mathbf{r}'(t)$.

(a) Show that

$$\frac{Df}{Dt} := \frac{d}{dt} f(\mathbf{r}(t), t) = \frac{\partial f}{\partial t} + \nabla f \cdot \mathbf{v}.$$

Such a derivative is commonly referred to in fluid dynamics as the material (or convective) derivative and measures the rate of change along a moving path of some physical quantity which is being transported by fluid currents.

By the chain rule, we have

$$\frac{d}{dt} f(x(t), y(t), z(t), t) = f_t + f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = f_t + \nabla f \cdot \mathbf{r}'(t) = f_t + \nabla f \cdot \mathbf{v}(t).$$

(b) Let ρ be the density of the fluid. A fluid flow is said to be *incompressible* if

$$\frac{D\rho}{Dt} = 0.$$

Suppose further that the density depends only on the space variables (x, y, z) but not (explicitly) on t so that $\rho = \rho(x, y, z)$. An incompressible flow in this case is called *stratified*.

Show that $\nabla\rho \cdot \mathbf{v} = 0$ for stratified flow and interpret this condition.

From (a), we have

$$0 = \frac{D\rho}{Dt} = \rho_t + \nabla\rho \cdot \mathbf{v} = \nabla\rho \cdot \mathbf{v}.$$

This condition states that the directional derivative of the density in the direction $\mathbf{v}$ is zero, i.e., the density of the fluid does not change along the flow.

(c) A flow is called *steady* if the density ρ and the velocity field $\mathbf{v}$ do not explicitly depend on t, i.e, $\rho = \rho(x, y, z)$ and $\mathbf{v} = \mathbf{v}(x, y, z)$. In this case, the term *streamlines* is used for the paths of the particles in the flow since they keep their shape over time.

Suppose one has a 2D stratified steady flow so that $\rho = \rho(x, y)$ and $\mathbf{v} = \mathbf{v}(x, y)$ and suppose also that the density varies only by the height y. Draw a picture of the streamlines for such a flow and explain why the term "stratified" makes sense.

Since ρ varies only with y, we have $\nabla\rho = (0, \rho_y)$. From (b), we have $\nabla\rho \cdot \mathbf{v} = 0$ so the velocity vectors must be completely horizontal. Thus, the streamlines are parallel to the x-axis as in Figure 1.

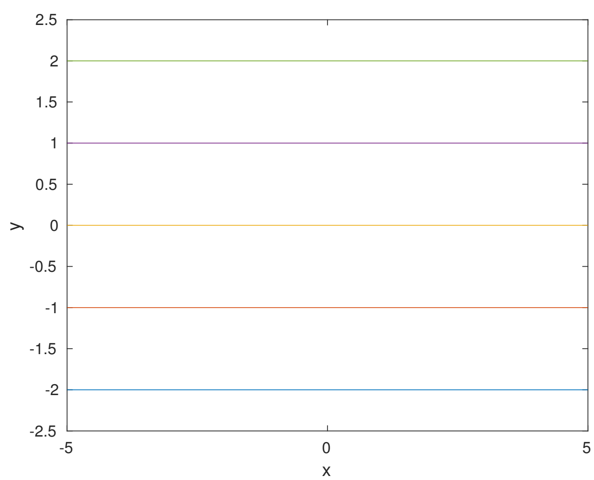

Figure 1

From (b), we know that the density of the fluid stays the same along the streamlines so the fluid is layered by density, i.e., stratified.